Combinatorics

Semestral Examination

Instructions: All questions carry ten marks.

1. Let $\mathcal{D}$ be a $3-(v,k,\lambda)$ design such that its derived design at a point is a symmetric design. Then show that
   (a) $\lambda(v-2)=(k-1)(k-2)$ and
   (b) any two blocks of $\mathcal{D}$ meet in 0 or $\lambda+1$ points.

2. Let $PG^2(4)$ denote the projective plane of order 4. Let $S=\{P_1,P_2,P_3,P_4\}$ be a subset of points of this plane such that no three of P_i's are collinear. Prove that there exists a unique set $\mathcal{O}$ of six points that contains S and no three points of $\mathcal{O}$ are collinear.

3. Let a,d be natural numbers. If $b=\lceil\frac{a}{d}\rceil$ or $\lfloor\frac{a}{d}\rfloor$, then prove that
$$\lceil\frac{a-b}{d-1}\rceil \le \lceil\frac{a}{d}\rceil \quad\text{and}\quad \lfloor\frac{a-b}{d-1}\rfloor \ge \lfloor\frac{a}{d}\rfloor.$$

4. Let G denote the group $\mathbb{Z}/21\mathbb{Z}$. Show that there exists a subset $S\subset G$ of cardinality 5 such that $2S=S$ and the collection $\mathcal{B}=\{S+a \mid a\in G\}$ form the projective plane of order 4, with elements of G as points.